BOUNDARY-LAYER THEORY APPLIED TO HIGH INJECTION RATES

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The equations of an isothermal laminar multicomponent boundary layer are solved numerically for both velocity and concentration profiles at high injection rates. The results are evaluated comparatively.

We consider the isothermal flow of an incompressible multicomponent gas in a laminar boundary layer with a negative pressure gradient and a high injection number $\Psi \sim \sqrt{\mathrm{Re}}(\rho v)_w/(\rho v)_\infty$. The stream in the boundary layer with heavy distributed injection can be divided into two regions [1-7]: 1) the inner region adjacent to the solid surface, where viscosity effect are negligible (to the first approximation); and 2) the "uplifted" viscous region with a transition from the inner-layer flow mode to the ideal-gas flow mode outside the boundary layer.

Self-adjoint flow modes were studied in [1-5]. Nonself-adjoint solutions to the boundary-layer equations were analyzed in [6, 7]. In [6], moreover, an asymptotic solution has been obtained to the Prandtl equation for a homogeneous incompressible fluid. Asymptotic formulas have also been obtained in [7] for the friction coefficient, for the thermal flux and the diffusion current in the fluid mixture components at the solid surface, and equations have been derived describing the flow of a compressible multicomponent gas in the sublayer region of a boundary layer. In our study here these equations will be solved for the isothermal flow of a multicomponent gas. A numerical solution will be given to the problem of gas flow in boundary layers of a sphere and of a circular cylinder.

1. The equations of a laminar isothermal multicomponent boundary layer, in the Dorodnitsyn–Lies variables

$$\xi = \int_0^x \mu_e \rho_e u_e r^{2k} dx, \quad \eta = \frac{u_e r^k}{\sqrt{2\xi}} \int_0^y \rho dy$$

are [7]:

$$\begin{gathered} (l f''_{\eta\eta})'_\eta + f f''_{\eta\eta} + \beta\,(\rho_e/\rho - f'^2_\eta) = 2\xi\,(f'_\eta f''_{\xi\eta} - f'_\xi f''_{\eta\eta}), \\ X'_{i\eta} = (f + 2\xi f'_\xi)\, c'_{i\eta} - 2\xi f'_\eta c'_{i\xi} \quad (i = 1, \ldots, n-1), \\ l\,(c_i m)'_\eta = \sum_{j=1}^{n} \frac{m^2}{m_j} S_{ij}\,(c_i X_j - c_j X_i) \quad (i = 1, \ldots, n-1), \\ \sum_{i=1}^{n} c_i = 0, \quad \sum_{i=1}^{n} X_i = 0. \end{gathered} \tag{1.1}$$

Here

$$\rho v = - r^{-k} \left[(f + 2\xi f'_\xi)\, \xi'_x + 2\xi f'_\eta \eta'_x\right] / \sqrt{2\xi};$$

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$$f'_\eta = \frac{u}{u_e}; \quad X_i = \frac{r^k \sqrt{2\xi}}{\xi'_x} I_i; \quad l = \frac{\mu\rho}{\mu_e\rho_e}; \quad \beta = \frac{2\xi}{u_e} \cdot \frac{du_e}{d\xi}.$$

System (1.1) is solved for the following boundary conditions:

$$f'_\eta \to 1, \ c_i \to c_{ie} (i = 1, \ldots, n) \ \text{as} \ \eta \to \infty,$$
$$f + 2\xi f'_\xi = \Phi(\xi), \ f'_\eta = 0,$$
$$\Phi(\xi)(c_i - c_i^{(1)}) = X_i \ (i = 1, \ldots, n) \quad \text{for} \ \eta = 0. \tag{1.2}$$

Here $c_i^{(1)}$ denotes the mass concentration of the i-th component in the injected gas mixture and

$$\Phi(\xi) = -\frac{r^k \sqrt{2\xi}}{\xi'_x} (\rho v)_w. \tag{1.3}$$

Relation (1.3) can be rewritten as

$$f(\xi, 0) = -\alpha(\xi) = -\frac{1}{\sqrt{2\xi}} \int_0^{x(\xi)} r^k (\rho v)_w \, dx. \tag{1.4}$$

2. We consider now the asymptotic solution to Eq. (1.1) at high values of the injection number ψ. It can be shown [7] that in this case near the solid surface Eqs. (1.1) reduce (in the first approximation) to

$$(\psi\varphi + 2\xi\varphi'_\xi)\varphi''_{\zeta\zeta} + \beta(\rho_e/\rho - \varphi'^2_\zeta) = 2\xi\varphi'_\zeta\varphi''_{\xi\zeta}, \tag{2.1}$$

$$(\psi\varphi + 2\xi\varphi'_\xi) c'_{i\zeta} = 2\xi\varphi'_\zeta c'_{i\xi} \quad (i = 1, \ldots, n-1) \tag{2.2}$$

with the boundary conditions at $\zeta = 0$

$$\varphi = 1, \ \varphi'_\zeta = 0, \ c_i = c_i^{(1)} \quad (i = 1, \ldots, n-1). \tag{2.3}$$

Here

$$\varphi = f/f(\xi, 0), \ \zeta = \eta/f(\xi, 0), \ \psi = \Phi(\xi)/f(\xi, 0). \tag{2.4}$$

It will be assumed further that the medium is incompressible and that the concentrations $c_i^{(1)}$ do not depend on ξ:

$$c_i^{(1)}(\xi) = \text{const} \quad (i = 1, \ldots, n-1). \tag{2.5}$$

The solution to Eqs. (2.2) with conditions (2.3) will then be

$$c_i(\xi, \zeta) = c_i^{(1)} \quad (i = 1, \ldots, n), \ \rho(\xi, \zeta) = \rho_w. \tag{2.6}$$

Let us now analyze Eq. (2.1). We introduce here new variables

$$Z = \varphi'^2_\zeta \rho_w/\rho_e, \quad \varphi = \varphi(\xi, \zeta). \tag{2.7}$$

Then Eq. (2.1) and conditions (2.3), with (2.5) taken into account, become

$$\psi\varphi Z'_\varphi - 2\xi Z'_\xi = 2\beta(Z - 1), \tag{2.8}$$

$$Z = 0 \quad \text{for} \ \varphi = 1. \tag{2.9}$$

It is not difficult to see that function Z has the simple physical meaning:

$$Z = \rho_w u^2/\rho_e u_e^2. \tag{2.10}$$

The solution to Eq. (2.8) with condition (2.9) is

$$Z = 1 - \exp\left(\int_\xi^\tau \frac{\beta}{\xi'} d\xi'\right). \tag{2.11}$$

Here function $\tau(\xi, \varphi)$ is determined from the equation

$$\xi\alpha^2(\xi)\varphi^2 = \tau\alpha^2(\tau), \tag{2.12}$$

where function α, in turn, is defined by expression (1.4).

Expressing β in terms of ξ and u_e again, we rewrite solution (2.11) as follows:

$$Z = 1 - u_e^2 \{\tau [\xi\alpha^2(\xi)\varphi^2]\}/u_e^2(\xi). \tag{2.13}$$

With the aid of relations (2.4), (2.7), and (2.13), we obtain

$$\eta = \alpha(\xi)(\rho_e/\rho_w)^{-1/2} \int_1^{\varphi} \frac{d\varphi}{\sqrt{Z}}. \tag{2.14}$$

Here φ varies over the range $0 \le \varphi \le 1$.

The equation of the flow separatrix is obtained by letting $\varphi = 0$ at the upper limit of the integral:

$$\eta^*(\xi) = \alpha(\xi)(\rho_e/\rho_w)^{-\frac{1}{2}}\Delta(\xi), \quad \Delta(\xi) = \int_1^0 \frac{d\varphi}{\sqrt{Z(\xi, \varphi)}}. \tag{2.15}$$

3. For illustration, we will consider the solution for the isothermal gas flow in the boundary layer of a circular cylinder and a sphere: $u_e = [(k+2)/(k+1)]U_\infty \sin\theta$.

a) When a cylinder is immersed in the stream (k = 0), then

$$u_e^{0^2} = \xi^0 - \xi^{0^2}, \quad \xi^0 = 1/2(1 - \cos\theta), \quad u_e^0 = u_e/4U_\infty. \tag{3.1}$$

1°. If $(\rho v)_w$ = const, then

$$\begin{gathered} f(\xi, 0) = -\alpha(\xi) = -\alpha(0)[\arccos(1 - 2\xi^0)]/\sqrt{2\xi^0}, \\ \alpha(0) = \sqrt{\mathrm{Re}}\,(\rho v)_w/2(\rho U)_\infty, \quad \mathrm{Re} = 2U_\infty R/\nu_\infty. \end{gathered} \tag{3.2}$$

Inserting the first expressions in (3.1) and in (3.2) into (2.12) and then solving for τ will yield

$$\tau = 1/2\{1 - \cos[\varphi \arccos(1 - 2\xi^0)]\}. \tag{3.3}$$

Formula (2.13) with (3.3) becomes then

$$Z = 1 - 1/4\,\frac{\sin^2[\varphi \arccos(1 - 2\xi^0)]}{\xi^0 - \xi^{0^2}}. \tag{3.4}$$

Inserting (3.4) into (2.14) yields coordinate η as a function of φ.

2°. If the specific flow rate of gas through the surface is distributed so that $f(\xi, 0)$ = const, ($(\rho v)_w = -2f(\xi, 0)(\rho U)_\infty \cos(\theta/2)\mathrm{Re}^{-1/2}$), then (2.13) can be written as

$$Z = 1 - \varphi^2(1 - \xi^0\varphi^2)/(1 - \xi^0). \tag{3.5}$$

b) When a sphere is immersed (k = 1), then we have

$$\begin{gathered} u_e^{0^2} = 9/4\{1 - 4\cos^2 1/3[\arccos(\xi^0 - 1) + 4\pi]\}, \\ \xi^0 = 1 + 1/2\cos\theta(\cos^2\theta - 3), \quad u_e^0 = u_e\Big/\frac{3}{2}U_\infty, \\ Z = 1 - \frac{1 - 4\cos^2\{1/3[\arccos(\tau - 1) + 4\pi]\}}{1 - 4\cos^2\{1/3[\arccos(\xi^0 - 1) + 4\pi]\}}. \end{gathered} \tag{3.6}$$

Here for $f(\xi, 0)$ we have

$$\tau = \xi^0\varphi^2 \tag{3.7}$$

and, if $(\rho v)_w$ = const,

$$\tau = 1 + \cos 3\arccos 1/2\{1 + \varphi[2\cos 1/3\{\arccos(\xi^0 - 1) + 4\pi\} - 1]\}. \tag{3.8}$$

Thus, expressions (2.14) and (3.4) or (3.5) for an immersed cylinder and expressions (2.14), (3.6), and (3.7) or (3.8) for an immersed sphere will completely define the flow in the sublayer region of a boundary layer.

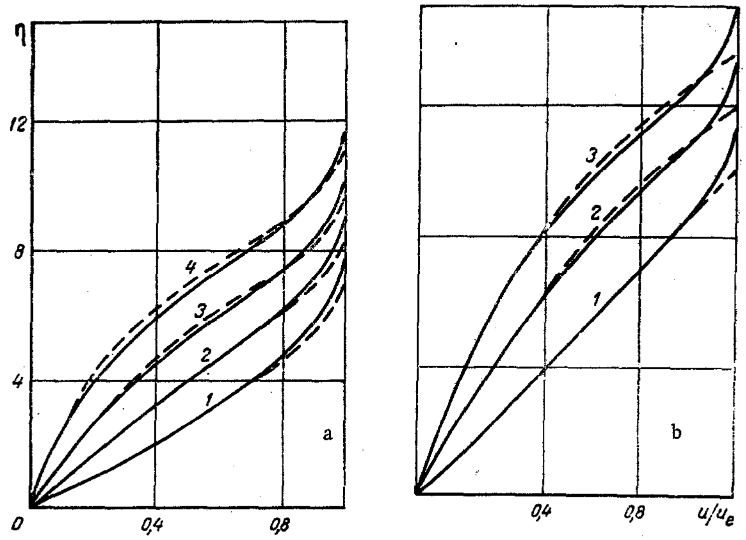

Fig. 1. Velocity profile u/u_e across the boundary layer: a) around a cylinder at four sections [1) $\xi = 0$; 2) 0.3; 3) 0.4; 4) 0.45]; b) around a sphere [1) $\theta = 0$; 2) 1.25; 3) 1.41 rad]. Injection number $\alpha = 5$, $\rho = \rho_e = \rho_w$. Solid lines) Numerical solution; dashed lines) asymptotic solution.

4. Equations (1.1) with the boundary conditions (1.2) were solved by the numerical method in [7] with the aid of a digital computer. Considered were: 1) flow of a homogeneous incompressible fluid in the boundary layers of a sphere and a circular cylinder, respectively; and 2) isothermal flow of a gas mixture containing hydrogen, nitrogen, and carbon dioxide in the boundary layer of a sphere. In both cases the injection number was assumed constant and its value was varied over the range $0 \leq \alpha \leq 10$ for different sets of computations.

Furthermore, in the second case it was assumed that the concentrations of injected H_2, N_2, and CO_2 gases were constant along the generatrix of the sphere ($c_i^{(1)}$ = const).

In Fig. 1a are shown velocity profiles u/u_e across the boundary layer around a cylinder at four sections. It is noteworthy that the velocity profiles calculated numerically and those based on the asymptotic solution overlap almost across the entire boundary layer when the injection number is $\alpha \gtrsim 7$.

In Fig. 1b are shown analogous velocity profiles across the boundary layer of a sphere at three sections: $\theta = 0$, 1.25, and 1.41 rad (curves 1, 2, 3, respectively).

Velocity profiles u/u_e and density profiles ρ/ρ_e across the boundary layer of a sphere immersed in an $H_2 + N_2 + CO_2$ mixture are shown in Fig. 2 at three sections: $\theta = 0$, 1.25, and 1.41 rad (curves 1, 2, 3,

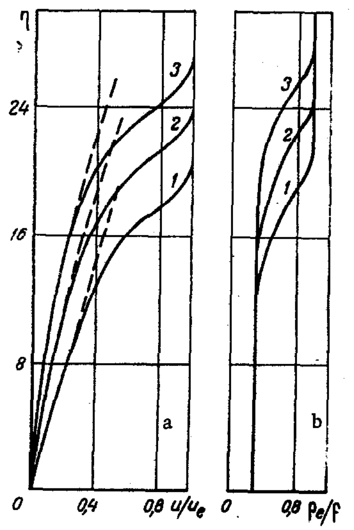

Fig. 2. a) Velocity profiles u/u_e; and b) density profiles ρ_e/ρ across the boundary layer of a sphere immersed in an $H_2 + N_2 + CO_2$ gas stream, with $\alpha = 5$ and $\rho_e/\rho_w = 0.28$: 1) $\theta = 0$; 2) 1.25; 3) 1.41 rad. Solid lines) Solution; dashed lines) asymptotic solution.

respectively). It is to be noted that the concentrations of the gas components at the solid surface become approximately equal to their concentrations in the injected mixture ($c_{iw} \approx c_i^{(1)}$) when the injection number is within the range $\alpha = 2$–3. As the injection number increases, near the solid surface there will appear a region where $c_i(\xi, \eta) \approx c_{iw}$. Here the velocity profile based on the asymptotic solution will overlap with the velocity profile calculated numerically. The width of this region is accurately enough determined from formula (2.15).

NOTATION

x, y	are the coordinates, along the body surface and normal to the body surface, respectively;
u, v	are the velocity components;
r	is the distance from the symmetry axis of the body;
$k = 0$	is for two-dimensional flow;
$k = 1$	is for flow with axial symmetry;
ρ	is the density of gas;
μ	is the dynamic viscosity of gas;
m	is the molecular weight of gas;
c_i	is the mass concentration of i-th component;
I_i	is the diffusive mass current of i-th component;
m_i	is the molecular weight of i-th component;
D_{ij}	is the binary diffusion coefficient;
S_{ij}	is the Schmidt number;
ξ, η	are the Dorodnitsyn–Lies variables;

$f'_\eta = u/u_e$;

$X_i = (r^k\sqrt{2\xi}/\xi'_x)I_i$;

$l = \mu\rho/\mu_e\rho_e$;

$\beta = (2\xi/u_e)(du_e/d\xi)$;

$\Phi(\xi) = -(r^k\sqrt{2\xi}/\xi'_x)(\rho v)_w$.

Subscripts

e denotes the outside edge of the boundary layer;

w denotes the solid surface (wall).

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